THE CHINESE UNIVERSITY OF HONG KONG
MATH3270B
MIDTERM SOLUTION
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Question 1:

1. It's a separable equation. We deduce that $(3y^2 - 6)y' = 3t^2$, integral on both sides and use the initial data, we deduce the solution is

$$y^3 - 6y = t - 1$$

.

2. By the hint, we want to find an integrating factor having the form $u = u(xy)$. From the pro 5 in homework1, we can choose $u = e^{\int R}$, where $R = \frac{N_x - M_y}{xM - yN}$, which only depends on xy. In this equation, we find that $N_x = 1 + 4xy$, $M_y = 3 + 4xy$, and then $R = \frac{N_x - M_y}{xM - yN} = -\frac{1}{xy}$(only depends on xy). Hence, $u = \frac{1}{xy}$. We multiply u on the both sides and get an exact equation $(\frac{3}{x} + 2y)dx + (\frac{1}{y} + 2x)dy = 0$. Therefore, using the initial data, the solution of that O.D.E is

$$\ln|y| + 2xy + 3\ln|x| = 2.$$

3. The characteristic equation of the homogeneous equation is $4r^2 - 4r + 1 = (2r - 1)^2 = 0$. Hence, the general solution to the homogeneous equation is $y = (c_1 + c_2 t)e^{t/2}$. Since the R.H.S of inhomogeneous equation is $8e^{t/2}$, we assume the particular solution having the form $y^* = At^2 e^{t/2}$. Then, $(y^*)' = (\frac{At^2}{2} + 2At)e^{t/2}$, $(y^*)'' = (\frac{At^2}{4} + 2At + 2A)e^{t/2}$, we substitute these to the equation and deduce that $A = 1$. Hence, the solution to the O.D.E is

$$y = (c_1 + c_2 t)e^{t/2} + t^2 e^{t/2}.$$

Question 2:

1. We consider the general case. Suppose the equation $p_2 y'' + p_1 y' + p_0 y = 0$ has a solution φ_1, we assume another linearly independent solution is $y = v(t)\varphi_1$ and calculate each derivative of y:

$$\begin{aligned} y' &= v'\varphi_1 + \varphi_1' v, \\ y'' &= v''\varphi_1 + 2v'\varphi_1' + v\varphi_1''. \end{aligned} \tag{1}$$

[1] In the marking of midterm, we will give 50%-80% marks if your result is wrong but the process is right.

We substitute above to the general equation, using the fact that φ_1 is a solution, and we will get v satisfies

$$p_2\varphi_1 v'' + (2p_2\varphi_1' + p_1\varphi_1)v' = 0. \tag{2}$$

In this problem, $\varphi_1 = e^t$, $p_2 = \sin t$, $p_1 = -(\sin t + \cos t)$, we substitute these to (2) and get that v satisfies

$$\sin t v'' + (\sin t - \cos t)v' = 0. \tag{3}$$

Let $w = v'$ and we deduce that $w = \frac{\sin t}{e^t}$, hence, $v(t) = -e^{-t}\frac{\sin t + \cos t}{2}$. Now we deduce another solution is $y_2 = -\frac{\sin t + \cos t}{2}$, so the general solution to the O.D.E is $y = c_1 e^t + c_2 \frac{\sin t + \cos t}{2}$.

2. We also consider the general case. Suppose the equation $p_3 y^{(3)} + p_2 y'' + p_1 y' + p_0 y = 0$ has a solution φ_1, we assume another linearly independent solution is $y = v(t)\varphi_1$ and calculate each derivative of y:

$$\begin{aligned} y' &= v'\varphi_1 + \varphi_1' v, \\ y'' &= v''\varphi_1 + 2v'\varphi_1' + v\varphi_1'', \\ y^{(3)} &= v'''\varphi_1 + 3v''\varphi_1' + 3v'\varphi_1'' + v\varphi_1'''. \end{aligned} \tag{4}$$

We substitute above to the general equation, using the fact that φ_1 is a solution, and we will get v satisfies

$$p_3\varphi_1 v''' + (3p_3\varphi_1' + p_2\varphi_1)v'' + (3p_3\varphi_1'' + 2p_2\varphi_1' + p_1\varphi_1)v' = 0. \tag{5}$$

In this problem, $p_3 = t^2 - 2t + 2$, $p_2 = -t^2$, $p_1 = 2t$ and we choose $\varphi_1 = e^t$. Similarly, we get that v satisfies

$$(t^2 - 2t + 2)v''' + (2t^2 - 6t + 6)v'' + (t^2 - 4t + 6)v' = 0. \tag{6}$$

By far we can't solve this equation, but since $\varphi_2 = t$ is solution to the O.D.E, which means $v_1 = te^{-t}$ is a particular solution to (6). Hence, we use the reduction of order again. Let $w = v'$ and assume $w = uv_1'$ is another solution to the following equation:

$$(t^2 - 2t + 2)w'' + (2t^2 - 6t + 6)w' + (t^2 - 4t + 6)w = 0. \tag{7}$$

Use the formula in 3.1, we deduce that u satisfies:

$$t(t^2 - 2t + 2)u'' = 0. \tag{8}$$

Then, $u = C_1 + C_2 t$, which means $v = C_1 te^{-t} + C_2(t^2 + t - 1)e^{-t}$. Hence, another linearly independent solution to the O.D.E is $\varphi_3 = t^2$, which means the general solution is $y = c_1 t + c_2 t^2 + c_3 e^t$.

REMARK: In this problem, we can also choose $\varphi_1 = t$, this leads to the equation of v changing to

$$t(t^2 - 2t + 2)v''' + (-t^3 + 3t^2 - 6t + 6)v'' = 0. \tag{9}$$

Actually, (9) is a first order separable equation, but solving this equation is not easy. We let $w = v''$ and use partial fraction to find that $w = e^t \frac{(t-1)^2+1}{t^3}$, then integral twice to deduce the same result.(The calculation is not easy and you can have a try by yourself.)

Question 3:

1. For this question, we add the condition that **THE WRONSKIAN OF $\boldsymbol{\varphi_1, \varphi_2 ... \varphi_n}$ IS NONZERO**. We consider the following system of linear equations:

$$\begin{cases} \varphi_1^{(n-1)} p_{n-1}(t) + ... + \varphi_1' p_1(t) + \varphi_1 p_0(t) = -\varphi_1^{(n)}(t) \\ \varphi_2^{(n-1)} p_{n-1}(t) + ... + \varphi_2' p_1(t) + \varphi_2 p_0(t) = -\varphi_2^{(n)}(t) \\ ...... \\ \varphi_n^{(n-1)} p_{n-1}(t) + ... + \varphi_n' p_1(t) + \varphi_n p_0(t) = -\varphi_n^{(n)}(t) \end{cases} \tag{10}$$

We write the (10) in the vector form: $Ap = b$, where $A = \begin{pmatrix} \varphi_1 & \varphi_1' & \cdots & \varphi_1^{(n-1)} \\ \varphi_2 & \varphi_2' & \cdots & \varphi_2^{(n-1)} \\ \cdots & & & \\ \varphi_n & \varphi_n' & \cdots & \varphi_n^{(n-1)} \end{pmatrix}$, $p = (p_0, p_1, ..., p_{n-1})^T$, $b = (-\varphi_1^{(n)}, -\varphi_2^{(n)}, ..., -\varphi_n^{(n)})^T$. Now since $A = W(\varphi_1, \varphi_2, ..., \varphi_n)^T$, which implies $det(A) = det(W) \neq 0$. Hence, the system (10) is solvable, we can apply the Cramer's rule to solve these p_i. The Cramer's rule tells us $p_i = \frac{det(W_i)}{det(W)}$, where W_i is the matrix formed by replacing the i-th column of A by the column vector b. The remaining thing is that to show these coefficients are continuous. Since each φ_i is n-times continuous differentiable functions, and the determinant of n^2 elements is a polynomial of these elements, we deduce that W_i and W are continuous functions, which implies these p_i are continuous.

2. From 3.1, we have the formula $p_i = \dfrac{det(W_i)}{det(W)}$. In this question, we have:

$$\begin{aligned} W(e^t, \sin t) &= \begin{vmatrix} e^t & \sin t \\ e^t & \cos t \end{vmatrix} = e^t(\cos t - \sin t), \\ W_0 &= \begin{vmatrix} e^t & \cos t \\ -e^t & \sin t \end{vmatrix} = e^t(\sin t + \cos t), \\ W_1 &= \begin{vmatrix} e^t & \sin t \\ -e^t & \sin t \end{vmatrix} = 2e^t \sin t. \end{aligned} \tag{11}$$

Hence we deduce that $p_0 = -\dfrac{\sin t + \cos t}{\cos t - \sin t}$, $p_1 = \dfrac{2\sin t}{\cos t - \sin t}$, the equation is $y'' + p_1 y' + p_0 y = 0$.

REMARK: In this problem, note that p_{n-1} can also be solved by Abel's formula.

Question 4:

1. Since:

$$|\lambda I - A| = \begin{vmatrix} \lambda & 3 & -1 & -2 \\ 2 & \lambda-1 & 1 & -2 \\ 2 & -1 & \lambda+1 & -2 \\ 2 & 3 & -1 & \lambda-4 \end{vmatrix} = \begin{vmatrix} \lambda & 3 & -1 & -2 \\ 2 & \lambda-1 & 1 & -2 \\ 0 & 0 & \lambda & 0 \\ 0 & 2-\lambda & -2 & \lambda-2 \end{vmatrix} = \lambda \begin{vmatrix} \lambda & 2 & -2 \\ 2 & \lambda & -2 \\ 0 & 2-\lambda & \lambda-2 \end{vmatrix}$$
$$= \lambda(\lambda-2) \begin{vmatrix} \lambda & 0 \\ 2 & \lambda-2 \end{vmatrix} = \lambda^2(\lambda-2)^2. \tag{12}$$

Hence, we deduce that A has two double eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2$ and the corresponding eigenvectors are $\xi = (-1,-1,-1,-1)^T$, $\eta_1 = (0,-1,-1,-1)^T$, $\eta_2 = (3,1,1,4)^T$. Hence the Jordan form of A is $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = D$. Since $dim(ker(A - \lambda_1 I)) = 1$, $dim(ker(A - \lambda_1 I)^2) = 2$, we want to find β such that $\beta \in ker(A - \lambda_1 I)^2 / ker(A - \lambda_1 I)$. We calculate that $A^2 = \begin{pmatrix} 0 & -8 & 4 & 4 \\ -4 & 0 & 0 & 4 \\ -4 & 0 & 0 & 4 \\ -4 & -8 & 4 & 8 \end{pmatrix}$, so we can choose $\beta = (1,2,3,1)^T$. By far, we deduce that $T = \begin{pmatrix} -1 & 1 & 0 & 3 \\ -1 & 2 & -1 & 1 \\ -1 & 3 & -1 & 1 \\ -1 & 1 & -1 & 4 \end{pmatrix}$, where $T^{-1}AT = D$.

2. Since the Jordan form of A is D, and $exp(Dt) = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & 0 & e^{2t} \end{pmatrix}$. We get that the fundamental matrix $\varphi(t) = Te^{Dt}$. Hence the general solution is

$$x = c_1 \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} + c_2 t \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix} + c_4 e^{2t} \begin{pmatrix} 3 \\ 1 \\ 1 \\ 4 \end{pmatrix} \tag{13}$$

REMARK: In this problem, since we have three eigenvectors, we immediately have three solutions: $x_1 = \xi$, $x_3 = \eta_1 e^{2t}$, $x_4 = \eta_2 e^{2t}$. For another solution (I have already told in the tutorial), we can assume it having the form $x_2 = \xi t + \beta$, where β satisfies $(A - \lambda_1 I)\beta = \xi$.

Question 5:

1. It's sufficient to check that $x_1 = (3t^4, -t^4)^T$, $x_2 = (t^2, -t^2)^T$ are the solutions of that system(Obviously they are linearly independent). We calculate that

$$P(t)x_1 = \begin{pmatrix} 12t^3 \\ -4t^3 \end{pmatrix} = x_1^{'}, \tag{14}$$

$$P(t)x_2 = \begin{pmatrix} 2t \\ -2t \end{pmatrix} = x_2^{'}. \tag{15}$$

These complete our proof.

2. Since the fundamental matrix of homogeneous system is $\varphi(t)$, we assume the solution to the O.D.E is $x = u(t)\varphi(t)$, where $u = (u_1, u_2)^T$. The variation of parameters tells us u satisfies

$$\begin{pmatrix} 3t^4 & t^2 \\ -t^4 & -t^2 \end{pmatrix} \begin{pmatrix} u_1^{'} \\ u_2^{'} \end{pmatrix} = \begin{pmatrix} 4t^4 \\ 0 \end{pmatrix}, \tag{16}$$

i.e.

$$\begin{cases} 3t^4 u_1^{'} + t^2 u_2^{'} = 4t^4 \\ -t^4 u_1^{'} - t^2 u_2^{'} = 0 \end{cases} \tag{17}$$

We solve (17) and deduce that $u_1 = 2t + C_1$, $u_2 = -\frac{2}{3}t^3 + C_2$. Therefore, the general solution of that O.D.E is

$$x = C_1 t^4 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + C_2 t^2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + t^5 \begin{pmatrix} \frac{16}{3} \\ -\frac{4}{3} \end{pmatrix}. \tag{18}$$

Question 6: This problem is a simple use of Abel's formula. In tutorial, I already showed that for the n-th order linear equation:

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + ... + p_1(t)y' + p_0 y = 0, \tag{19}$$

we have $W(y_1, ...y_n) = Ce^{\int -p_{n-1}(t)dt}$, where $y_1, ...y_n$ are the solutions of (19). For the system $x' = Ax$, we have similar result. If $\varphi_1, ...\varphi_n$ are solutions of that systems, we have $\hat{W}(\varphi_1, ..., \varphi_n) = C' e^{\int tr(A)dt}$, where $tr(A)$ means the trace of A. In this problem, since the $tr(A) = -p(t)$ and $y_1, y_2, y_3, \varphi_1, \varphi_2, \varphi_3$ are fundamental sets, we deduce that

$$\begin{aligned} W(y_1, ..., y_3) &= C_1 e^{\int -p(t)dt} \\ \hat{W}(\varphi_1, ..., \varphi_3) &= C_2 e^{\int -p(t)dt} \end{aligned} \tag{20}$$

where C_1, C_2 are non-zero constants. This completes the proof.